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Light-cone formulation and spin spectra of massive strings

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Abstract

It is shown that all bosonic and fermionic massive string models admit consistent light-cone formulations. This result is used to derive the spin generating functions of these models in four dimensions.

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1 Introduction

The covariant quantization of the free Nambu-Goto string [1], and the Ramond-Neveu-Schwarz fermionic string [2, 3] leads in non-critical dimensions to consistent quantum models with longitudinal excitations. It was first pointed out by Polyakov [4, 5] that the dynamics of these extra degrees of freedom should be described by the (super)-Liouville theory. This suggests the modifications of the standard string world-sheet actions by adding the (super)-Liouville sectors. In the context of free string such modifications were first analyzed by Marnelius [6]. More recently the covariant quantization of these models was reconsidered under the assumption that the bulk and the boundary cosmological constants vanish [7, 8]. The analysis of the physical state conditions in terms of DDF operators exhibits three types of excitations corresponding to the transverse, the Liouville, and the longitudinal degrees of freedom. The first two are described by the standard Fock spaces while the latter one is given by a unitary highest weight representation of the (super)-Virasoro algebra [7, 8]. The classification of the ghost-free models is then related to the classification of the unitary (super)-Virasoro Verma modules [9], and yields one continuous, and one discrete series of models. In all cases the first excited state is massive which justifies the name *massive string* introduced in our previous papers.

One member of the discrete series is especially interesting. It corresponds to the (super)-algebra of longitudinal excitations with vanishing central charge. In this case the bosonic massive string is equivalent to the non-critical Nambu-Goto string [7], and the fermionic one to the non-critical RNS string [8]. Both models admit the quantum light-cone formulation which can be used to analyse their spin spectra [10, 11].

In the present letter we address the problem of the structure and the physical content of other quantum massive string models. Our main result is the light-cone formulation of all models. We present the derivation for the discrete series, but the same technique can be used for the continuous series as well. The light-cone formulation allows for a complete description of the spin content of the theory. This is done in terms of the spin generating functions for the bosonic and fermionic massive strings in four dimensions.

Our results show that all the massive string models have essentially the same structure. In contrast to the critical strings where the number of the tachion-free models is strongly limited, the GSO projection yields a large class of tachion-free massive strings with surprisingly rich mass and spin spectra. None of these models contains massless states with spin greater than 1. This makes them good candidates for matter fields in an effective description of low energy QCD. The central problem of such application is a consistent interaction. There are at least two possibilities: either the contact joining-splitting interaction, or the exchange of a vector particle to which the massive string naturally couples by its world-sheet action. The light-cone formulation provides a setting in which one can pose and in principle answer the question of the Lorentz covariance of both kinds of interactions. This was our main motivation for the present work.

The paper is divided in two parts. The first one is devoted to the bosonic models and contains more detailed derivations. The second part concerns the fermionic strings.

2 Bosonic string

We define the light-cone massive bosonic string as a representation of the algebra

$$
\begin{aligned}
[a^i_0, q^j_0] &= -i\delta^{ij} \quad , \quad [a^+_0, q^-_0] = i \quad , \quad [c_0, q^L_0] = -i \quad , \\
[a^i_m, a^j_n] &= m\delta^{ij}\delta_{m,-n} \quad , \quad [c_m, c_n] = m\delta_{m,-n} \quad , \\
[L^{\rm L}_m, L^{\rm L}_n] &= (m-n)L^{\rm L}_{m+n} + \tfrac{c^{\rm L}}{12}(m^3-m)\delta_{m,-n} \quad .
\end{aligned}
$$

All other commutators vanish and the standard conjugation properties are assumed.

The algebra of non-zero modes is by construction isomorphic to the (diagonalized) algebra of the DDF operators of the covariant approach [7]. In particular, the algebra of $L^{\rm L}_m$ corresponds to the Virasoro algebra of the shifted Brower longitudinal operators with the central charge $c^{\rm L}$. The value of $c^{\rm L}$ is restricted by the no-ghost theorem [7] to the range $1 \leq c^{\rm L} < 25-d$, or to the discrete series

$$
c^{\rm L} = c_m \equiv 1 - \frac{6}{m(m+1)} \quad , \quad m = 2, 3, ... \quad .
$$

Let us denote by $\mathcal{V}_m(p,q)$ the unitary Verma module of the Virasoro algebra with the central charge c_m and with the highest weight [9]

$$
h_m(p,q) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)} \quad , \quad 1 \leq p \leq m-1; \;\; 1 \leq q \leq p \quad .
$$

The space of states in the light-cone formulation is given by

$$
\mathcal{H}_m(p,q) = \int \frac{dp_+}{|p^+|} d^{d-2}\overline{p}\; \mathcal{F}(p^+, \overline{p}) \otimes \mathcal{V}_m(p,q) \quad ,
$$

where $\mathcal{F}(p^+, \overline{p})$ denotes the Fock space generated by the transverse and the Liouville excitations out of the unique vacuum state Ω satisfying

$$
\sqrt{\alpha} a^i_0\, \Omega = p^i\, \Omega \quad , \quad \sqrt{\alpha} a^+_0\, \Omega = p^+\, \Omega \quad , \quad c_0\, \Omega = 0 \quad .
$$

In order to construct a unitary realization of the Poincare algebra on $\mathcal{H}_m(p,q)$ we introduce

$$
\begin{aligned}
L^{\rm A}_n &= \tfrac{1}{2} \sum_{k=-\infty}^{+\infty} \sum_{i=1}^{D-2} : a^i_{-k} a^i_{n+k} : \\
L^{\rm C}_n &= \tfrac{1}{2} \sum_{k=-\infty}^{+\infty} : c_{-k} c_{n+k} : \; + 2i\sqrt{\beta} n c_n \; + \; 2\beta\delta_{n,0} \quad .
\end{aligned}
$$

The operators $L^{\rm T}_n \equiv L^{\rm A}_n + L^{\rm C}_n + L^{\rm L}_n$ form the Virasoro algebra with the central charge $c^{\rm T}_m = d + 48\beta - \frac{6}{m(m+1)}$. Following the standard light-cone construction we define the Poincare generators

$$
P^+ = \sqrt{\alpha} a^+_0 \quad , \quad P^i = \sqrt{\alpha} a^i_0 \quad , \quad P^- = \frac{\sqrt{\alpha}}{a^+_0}(L^{\rm T}_0 - \alpha_0) \quad ,
$$

$$
\begin{aligned}
M^{ij}_{\rm lc} &= P^i x^j - P^j x^i + i \sum_{n\geq 1} \frac{1}{n}(a^i_{-n}a^j_n - a^j_{-n}a^i_n) \quad , \\
M^{i+}_{\rm lc} &= P^+ x^i \quad , \\
M^{+-}_{\rm lc} &= \tfrac{1}{2}(P^+ x^- + x^- P^+) \quad , \\
M^{i-}_{\rm lc} &= \tfrac{1}{2}(x^i P^- + P^- x^i) - P^i x^- - i\frac{\sqrt{\alpha}}{P^+} \sum_{n\geq 1} \frac{1}{n}\left(a^i_{-n}L^{\rm T}_n - L^{\rm T}_{-n}a^i_n\right),
\end{aligned}
$$

It is known [10, 12] that the algebra of these generators closes to the Lie algebra of Poincare group if and only if the central charge $c^{\rm T}_m$ of the Virasoro algebra $\{L^{\rm T}_n\}$ and α_0 entering the definition of P^- take the critical values $c^{\rm T}_m = 24,\ \alpha_0 = 1$. This conclusion is independent of any detailed structure of $L^{\rm T}_n$ as far as the commutation relations

$$
[L^{\rm T}_m, a^i_n] = -n a^i_{m+n} \quad , \quad [L^{\rm T}_m, q^i_0] = -i a^i_m \quad ,
$$

are satisfied. Solving the condition $c^{\rm T}_m = 24$ for the parameter β one recovers the discrete series of values predicted by the no-ghost theorem of the covariant approach [7]:

$$
\beta = \beta_m \ = \ \frac{24-d}{48} + \frac{1}{8m(m+1)} \quad .
$$

Calculating the mass square operator one gets

$$
M^2 = 2\alpha\left(R_m(p,q) + \frac{((m+1)p - mq)^2}{4m(m+1)} - \frac{d}{24}\right) \tag{1}
$$

where $R_m(p,q)$ is the level operator in $\mathcal{H}_m(p,q)$.

For any $m \geq 2$ the space of states $\mathcal{H}_m(p,q)$ is by construction isomorphic to the space $\mathcal{H}^{\rm cov}_m(p,q)$ of physical states of the corresponding covariant massive string model. It follows from (1) that the mass spectra of these models coincide. In the case of $m = 2$ also the Poincare algebra representations are isomorphic [10]. In order to extend this result for all $m > 2$ we shall construct an isomorphism $\mathcal{I} : \mathcal{H}^{\rm cov}_m(p,q) \to \mathcal{H}_m(p,q)$. To this end it is convenient to work with the parameterization of $\mathcal{H}^{\rm cov}_m(p,q)$ in terms of the "original" (not diagonal) basis of the transverse A^i_n, the Liouville C_n, and the not "shifted" longitudinal $\widetilde{A}^{\scriptscriptstyle L}_n$ DDF operators. (For details concerning the DDF construction and the notation used in this section we refer to [10].) Since all states in $\mathcal{H}^{\rm cov}_m(p,q)$ are on mass-shell, x^+ can be regard as an evolution parameter in any representation in which the operator $\hat{x}^+$ is diagonal. Each physical state $|\psi\rangle \in \mathcal{H}^{\rm cov}_m(p,q)$ is then uniquely determined by its $x^+ = 0$ initial condition $\overline{|\psi\rangle}$, and each operator A on $\mathcal{H}^{\rm cov}_m(p,q)$ defines an operator $\overline{A}\,\overline{|\psi\rangle} \equiv \overline{A|\psi\rangle}$ acting on the space $\overline{\mathcal{H}}_m(p,q)$ of all $x^+ = 0$ initial conditions.

We define the isomorphism $\mathcal{I} : \mathcal{H}^{\rm cov}_m(p,q) \to \mathcal{H}_m(p,q)$ as a composition $\mathcal{I} = \mathcal{J} \circ \overline{\mathcal{I}}$, where $\overline{\mathcal{I}} : \mathcal{H}^{\rm cov}_m(p,q) \ni |\psi\rangle \to \overline{|\psi\rangle} \in \overline{\mathcal{H}}_m(p,q)$, and $\mathcal{J} : \overline{\mathcal{H}}_m(p,q) \to \mathcal{H}_m(p,q)$ is given by

$$
\mathcal{J}\overline{A}^i_n\mathcal{J}^{-1} = a^i_n \quad ; \quad \mathcal{J}\overline{\widetilde{A}}{}^{\scriptscriptstyle L}_n\mathcal{J}^{-1} = L^T_n \quad ; \quad \mathcal{J}\overline{C}_n\mathcal{J}^{-1} = c_n \quad . \tag{2}
$$

The Poincare algebra representations are equivalent if for all covariant Poincare generators $G_{\rm cov}$ and their light-cone counterparts $G_{\rm lc}$ the following relations hold

$$
\begin{aligned}
\mathcal{J}\overline{[G_{\rm cov},A^i_n]}\mathcal{J}^{-1} &= [G_{\rm lc},a^i_n] \quad , &(3)\\
\mathcal{J}\overline{[G_{\rm cov},\widetilde{A}^{\scriptscriptstyle L}_n]}\mathcal{J}^{-1} &= [G_{\rm lc},L^{\scriptscriptstyle T}_n] \quad , &(4)\\
\mathcal{J}\overline{[G_{\rm cov},C_n]}\mathcal{J}^{-1} &= [G_{\rm lc},c_n] \quad . &(5)
\end{aligned}
$$

The Poincare generators as well as the transverse A^i_n, and the longitudinal $\widetilde{A}^{\scriptscriptstyle L}_n$ operators are identical in all covaraint models. Also the light-cone commutators appearing on the right hand sides of the equations (3), and (4) coincide. Indeed, for all $m > 2$ the algebra of $L^{\scriptscriptstyle T}_n$, and a^i_n on $\mathcal{H}_m$ is isomorphic to the corresponding one on $\mathcal{H}_2$. Thus, since the equations (3,4) are satisfied for $m = 2$ [10] so they are for all $m > 2$.

The Liouville DDF operator C_n depends on m only via the constant β_m and similar arguments can be used to prove the third equation. It can be also directly calculated for all m by means of the leading term method [13]. The only non-vanishing commutator on the l.h.s. of (5) reads

$$
\begin{aligned}
\overline{[M^{i-}_{\rm cov},C_n]} &= -\alpha\frac{nx^i}{p^+}\overline{C}_n + \sqrt{\alpha}\frac{in}{p^+}\left(\sum_{m>0}\frac{1}{m}(\overline{A}^i_{-m}\overline{C}_{n+m} - \overline{A}^i_m\overline{C}_{n-m})\right)\\
&\quad -\sqrt{\alpha}\frac{2}{p^+}n\sqrt{\beta_m}\overline{A}^i_n \quad ,
\end{aligned}
$$

and after substitution (2) it reproduces the light-cone commutator $[M^{i-}_{\rm lc},c_n]$. This completes our proof that the light-cone models introduced in this section provide the light-cone formulations of the covariant models found in [7]. As the simplest application of this result we shall investigate the spin content of these models in four dimensions.

For every $m \geq 2$ there are $\frac{1}{2}m(m-1)$ ground states. For $m > 2$ there are always tachionic and massive ground states. Massless ground states are very rare. The first one appears at $m = 24, p = q = 20$ the next at $m = 242, p = q = 198$.

For all $m \geq 2$ and allowed p, q the first excited states are massive and the space $\mathcal{F}(p^+,\overline{p}) \otimes \mathcal{V}_m(p,q)$ carries a nonlinear representation of the little group SO(3) and a linear representation of its "transverse" subgroup SO(2). The generating function for SO(2) characters takes the form

$$
\chi^m_{p,q}(t,\varphi) = t^{-h_m(p,q)}\chi^m_{p,q}(t)\chi(t)\prod_{k\in\mathbb{N}}\frac{1}{1-2t^k\cos(\varphi)+t^{2k}} \tag{6}
$$

where $\mathbb{N}$ is the set of positive integers. The partition function

$$
\chi(t) = \prod_{k\in\mathbb{N}}(1-t^k)^{-1}
$$

counts states in the Liouville sector. The multiplicity of states in the longitudinal sector is given by the minimal model partition function [14]

$$
\chi^m_{p,q}(t) = \chi(t)\sum_{k\in\mathbb{Z}}\left(t^{\alpha^m_{p,q}(k)} - t^{\alpha^m_{p,-q}(k)}\right) \quad ,
$$

where

$$\alpha^m_{p,q}(k) = \frac{[2m(m+1)k - qm + p(m+1)]^2 - 1}{4m(m+1)} \quad .$$

Taking into account the decomposition $\chi^j(\varphi) = 1 + \sum_{k=1}^{j} 2\cos(k\varphi)$ of the irreducible SO(3) character $\chi^j(\varphi)$ of spin $j \in I\!N$ into irreducible characters of the SO(2) subgroup, one can obtain the SO(3) generating function expanding (6) directly in terms of $\chi^j(\varphi)$. Using formulae (5.3) and (5.4) of Ref.[15] one gets

$$\chi^m_{p,q}(t,\varphi) = \sum_{j \in I\!N_0} \chi^m_{p,q}(j,t)\chi^j(\varphi) \quad ,$$

where $I\!N_0$ is the set of non-negative integers and the spin j generating function $\chi^m_{p,q}(j,t)$ is given by

$$\chi^m_{p,q}(j,t) = t^{-h_m(p,q)}\chi^m_{p,q}(t)\chi(t)^3 \sum_{k \in I\!N} t^{\frac{k(k-1)}{2}}(-1)^{k-1}(1-t^k)^2 t^{kj} \quad .$$

The expression above provides a complete description of the SO(3) representations. Note that the full space reflection has a non-linear realization on the space $\mathcal{F}(p^+,\overline{p}) \otimes \mathcal{V}_m(p,q)$. This makes the analysis of the O(3) content of the model much more complicated.

The construction of the light-cone formulation of the closed string models proceeds along the standard lines by tensoring two open string spaces of states with an appropriate identification of the zero modes, and with the additional constraint $(L_0^{T\,(\mathrm{left})} - L_0^{T\,(\mathrm{right})})|\psi\rangle = 0$ imposed. For $m = 2,3,4$ the only states satisfying this constraint belong to the "diagonal" sectors $\mathcal{H}^{(\mathrm{left})}_m(p,q) \otimes \mathcal{H}^{(\mathrm{right})}_m(p,q)$. The spin content can be easily deduced from the diagonal part (i.e. all terms of the form $x^a y^a$) of $\chi^m_{p,q}(x,\varphi)\chi^m_{p,q}(y,\varphi)$.

For $m \geq 5$ one gets also non-diagonal solutions corresponding to pairs $(p,q) \neq (p',q')$ such that the difference $h^{(\mathrm{left})}_m(p,q) - h^{(\mathrm{right})}_m(p',q')$ is an integer number. For $m = 5$ such pairs are given by $h_5(4,1) - h_5(1,1) = 3$ and $h_5(3,1) - h_5(2,1) = 1$. The spin content of the corresponding models can be easily derived from modified products of generating functions. For instance in the case of $\mathcal{H}^{(\mathrm{left})}_5(4,1) \otimes \mathcal{H}^{(\mathrm{right})}_5(1,1)$ the spin content is given by the diagonal part of the product $\chi^5_{4,1}(x,\varphi)y^3\chi^5_{1,1}(y,\varphi)$. For $m > 5$ the number of non-diagonal models slowly increases with m.

3 Fermionic string

We define the discrete series of fermionic non-critical light-cone string models as representations of the algebras of the transverse

$$\begin{aligned} [a^i_0, q^j_0] &= -i\delta^{ij} \quad , \quad [a^+_0, q^-_0] = i \quad , \\ [a^i_m, a^j_n] &= m\delta^{ij}\delta_{m,-n} \quad , \quad \{b^i_r, b^j_s\} = \delta^{ij}\delta_{r,-s} \quad , \end{aligned}$$

the Liouville

$$[c_0, q^L_0] = -i \quad , \quad [c_m, c_n] = m\delta_{m,-n} \quad , \quad \{d_r, d_s\} = \delta_{r,-s} \quad ,$$

and the longitudinal

$$
\begin{aligned}
[L^{\mathrm{L}}_m, L^{\mathrm{L}}_n] &= (m-n)L^{\mathrm{L}}_{m+n} + \tfrac{\hat{c}^{\mathrm{L}}_m}{8}(m^3-m)\delta_{m,-n} \quad , \\
[L^{\mathrm{L}}_m, G^{\mathrm{L}}_r] &= (\tfrac{1}{2}m - r)G^{\mathrm{L}}_{m+r} \quad , \\
\{G^{\mathrm{L}}_r, G^{\mathrm{L}}_s\} &= 2L^{\mathrm{L}}_{r+s} + \tfrac{\hat{c}^{\mathrm{L}}_m}{2}(r^2 - \tfrac{1}{4})\delta_{r,-s}
\end{aligned}
$$

excitations, supplemented by the standard conjugation properties. In the formulae above $m, n \in \mathbb{Z}; r, s \in \mathbb{Z} + \frac{\epsilon}{2}$, and $\epsilon = 1$ corresponds to the Neveu-Schwarz sector, while $\epsilon = 0$ to the Ramond sector. The construction depends on the discrete parameter $m = 2, 3, ...$ via the central charge

$$
\hat{c}^{\mathrm{L}}_m = 1 - \frac{8}{m(m+2)}
$$

of the superconformal algebra of the longitudinal excitations. Let us denote by $\mathcal{V}^{\epsilon}_m(p,q)$ the unitary Verma module with the highest weight [9]

$$
h^{\epsilon}_m(p,q) = \frac{((m+2)p - mq)^2 - 4}{8m(m+2)} + \frac{1-\epsilon}{16}
$$

where for the Neveu-Schwarz sector ($\epsilon = 1$) $1 \leq q \leq p \leq m-1$, and $p - q$ is even. For the Ramond sector ($\epsilon = 0$) $p - q$ is odd, and $1 \leq q \leq p - 1$ for $1 \leq p \leq [\frac{1}{2}(m-1)]$, and $1 \leq q \leq p + 1$ for $[\frac{1}{2}(m+1)] \leq p \leq m-1$. We assume that $\mathcal{V}^0_m(p,q)$ is a representation of the extended Rammond algebra which includes a fermion chirality operator $(-1)^F$. The highest weight vectors are thus two-fold degenerate, unless $h^0_m(p,q) - \frac{\hat{c}^{\mathrm{L}}}{16} = 0$. Let $\mathcal{F}_\epsilon(p^+, \overline{p})$ be the Fock space generated by the algebra of non-zero transverse and Liouville excitations out of the vacuum state Ω_ϵ satisfying

$$
\sqrt{\alpha} a^i_0 \,\Omega_\epsilon = p^i \,\Omega_\epsilon \quad , \quad \sqrt{\alpha} a^+_0 \,\Omega_\epsilon = p^+ \,\Omega_\epsilon \quad , \quad c_0 \,\Omega_\epsilon = 0 \quad .
$$

In the Neveu-Schwarz sector ($\epsilon = 1$) the space of states of the light-cone model is defined by

$$
\mathcal{H}^1_m(r,s) = \int \frac{dp_+}{|p^+|} d^{d-2}\overline{p} \; \mathcal{F}^1(p^+, \overline{p}) \otimes \mathcal{V}^1_m(r,s) \quad .
$$

In the Rammond sector ($\epsilon = 0$) the fermionic zero modes b^i_0, d_0 along with the 0-level fermion chirality operator Γ^F form the real Euclidean Clifford algebra $\mathcal{C}(d,0)$. Let $D(d)$ be the irreducible representation of the complexified Clifford algebra $\mathcal{C}^C(d) = \mathcal{C}(d,0) \otimes \mathbb{C}$. We define the space of states in the Ramond sector ($\epsilon = 0$) by

$$
\mathcal{H}^0_m(p,q) = \int \frac{dp_+}{|p^+|} d^{d-2}\overline{p} \; \mathcal{F}^0(p^+, \overline{p}) \otimes D(d) \otimes \mathcal{V}^0_m(p,q) \quad ,
$$

where the tensor product of the operator algebras representations is determined by the requirement that the subspace of non-excited states provides an irreducible representation $D(d+1)$ of the complexified Clifford algebra $\mathcal{C}^C(d+1)$ generated by b^i_0, d_0, Γ^F and

G_0. For even d, $D(d+1) = D(d)$, and the construction of $\mathcal{H}^0_m(p,q)$ does not depend on whether the highest weight vector of $\mathcal{V}^0_m(p,q)$ is degenerate or not.

The construction of a unitary representation of the Poincare group is a straightforward generalization of the construction given in Ref. [11] for the fermionic critical ($m = 2$) massive string. We introduce the operators

$$
\begin{aligned}
L^{\rm A}_m &= \tfrac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{i=1}^{D-2} :a^i_{-n}a^i_{n+m}: + \tfrac{1}{2}\sum_{r\in\mathbb{Z}+\frac{\epsilon}{2}}\sum_{i=1}^{D-2} r :b^i_{-r}b^i_{r+m}: + (1-\epsilon)\tfrac{d-2}{16}\delta_{m,0} \quad , \\
L^{\rm C}_n &= \tfrac{1}{2}\sum_{n\in\mathbb{Z}} :c_{-n}c_{n+m}: + \tfrac{1}{2}\sum_{r\in\mathbb{Z}+\frac{\epsilon}{2}} r :d_{-r}d_{r+m}: +2i\sqrt{\beta}mc_m + [\tfrac{1-\epsilon}{16} + 2\beta]\delta_{m0} \;, \\
G^{\rm A}_r &= \sum_{n\in\mathbb{Z}}\sum_{i=1}^{D-2} a^i_{-n}b^i_{n+r} \quad , \\
G^{\rm C}_r &= \sum_{n\in\mathbb{Z}} c_{-n}d_{n+r} + 4i\sqrt{\beta}rd_r \quad .
\end{aligned}
$$

The operators $L^{\rm T}_m = L^{\rm A}_m + L^{\rm C}_m + L^{\rm L}_m$, and $G^{\rm T}_r = G^{\rm A}_r + G^{\rm C}_r + G^{\rm L}_r$ form an $N = 1$ superconformal algebra with the central charge $\hat{c}^{\rm T} = \hat{c}^{\rm L}_m + d - 1 + 32\beta$. The generators of the Poincare group are defined by

$$
\begin{aligned}
P^+ &= \sqrt{\alpha}a^+_0 \quad , \quad P^i = \sqrt{\alpha}a^i_0 \quad , \quad P^- = \frac{\alpha}{P^+}(L_0 - \alpha_0) \\
M^{ij} &= x^iP^j - x^jP^i - i\sum_{n>0}\frac{1}{n}(a^i_{-n}a^j_n - a^j_{-n}a^i_n) \\
&\quad + (1-\epsilon)ib^i_0b^j_0 - i\sum_{r>0}(b^i_{-r}b^j_r - b^j_{-r}b^i_r) \quad , \\
M^{i+} &= x^iP^+ \quad , \\
M^{+-} &= \tfrac{1}{2}(P^+x^- + x^-P^+) \quad , \\
M^{i-} &= \tfrac{1}{2}(P^-x^i + x^iP^-) - x^-P^i - \frac{i}{a^+_0}\sum_{n>0}\frac{1}{n}(a^i_{-n}L^{\rm T}_n - L^{\rm T}_{-n}a^i_n) \\
&\quad + (1-\epsilon)\frac{i}{a^+_0}b^i_0G^{\rm T}_0 - \frac{i}{a^+_0}\sum_{r>0}(b^i_{-r}G^{\rm T}_r - G^{\rm T}_{-r}b^i_r) \quad .
\end{aligned}
$$

This algebra closes if and only if $\hat{c}^{\rm T} = 8$ and $\alpha_0 = \frac{1}{2}$ [11, 12]. Solving these conditions one gets the discrete series of values of the parameter β

$$
\beta = \beta_m = \frac{8-d}{32} + \frac{1}{4m(m+2)} \quad , \quad m = 2, 3, \ldots \quad ,
$$

and the mass square operator

$$
M^2_\epsilon = 2\alpha\left(R^\epsilon_m(p,q) + \frac{((m+2)p - mq)^2}{8m(m+2)} - \frac{d}{16}\epsilon\right)
$$

where $R^\epsilon_m(p,q)$ is the level operator in $\mathcal{H}^\epsilon_m(p,q)$. This exactly coincides with the results of the covariant approach [8]. Following the method of the previous section and using

the results of Ref.[11] one can show that in each sector and for every admissible m, p, q the representations of the Poincare algebra in the light-cone model and in the covariant model are isomorphic. This completes the derivation of the light-cone formulation of the discrete series of the fermionic massive models obtained in [8]. We shall briefly consider their spin content in four dimensions.

In the Neveu-Schwarz sector all ground states are scalars. For $m = 2$ there is one tachionic ground state. For $m > 2$ there are always tachionic and massive ground states. The massless ground states are very rare, the first one appears at $m = 16, p = q = 12$, the next at $m = 98, p = q = 70$.

In the Ramond sector all ground states are massive, except the only massless cases of even m and $p = \frac{m}{2}, q = \frac{m}{2} + 1$. In the massive case the space of ground states carries two massive spin $\frac{1}{2}$ representations while in the massless case there are two pairs of left and right Weyl spinors.

The generating function for SO(2) characters (normalized to integer and half-integer powers) takes the form

$$\begin{aligned} \chi^{\epsilon,m}_{p,q}(t,\varphi) &= t^{-h^{\epsilon}_{m}(p,q)} \chi^{\epsilon,m}_{p,q}(t) \chi^{\epsilon}(t) \left(4 \cdot 2 \cos\frac{\varphi}{2}\right)^{1-\epsilon} \\ &\times \prod_{k \in I\!N} \frac{1}{1 - 2t^k \cos(\varphi) + t^{2k}} \prod_{r \in I\!N - \frac{\epsilon}{2}} (1 + 2t^r \cos(\varphi) + t^{2r}) \quad , \end{aligned}$$

where the partition function

$$\chi^{\epsilon}(t) = \prod_{k \in I\!N} \frac{1 + t^{k - \frac{\epsilon}{2}}}{1 - t^k}$$

counts states in the super-Liouville sector. The multiplicity of states in the longitudinal sector is given by the character of the unitary representation of the super-Virasoro algebra (not extended in the Ramond sector) [9]

$$\chi^{\epsilon,m}_{p,q}(t) = \chi^{\epsilon}(t) \sum_{k \in Z\!\!\!Z} \left(t^{\gamma^m_{p,q}(k)} - t^{\gamma^m_{-p,q}(k)} \right) \quad ,$$

where

$$\gamma^m_{p,q}(k) = \frac{[2m(m+2)k - p(m+2) + qm]^2 - 4}{8m(m+2)} \quad .$$

Using the techniques developed in Ref.[15] one can expand $\chi^{\epsilon,m}_{p,q}(t,\varphi)$ in terms of the irreducible SO(3) characters $\chi^j(\varphi)$

$$\chi^{\epsilon,m}_{p,q}(t,\varphi) = \sum_{j \in I\!N_0 + \frac{1-\epsilon}{2}} \chi^{\epsilon,m}_{p,q}(j,t) \chi^j(\varphi) \quad .$$

In the Ramond sector $j \in I\!N - \frac{1}{2}$, and the spin generating functions read

$$\begin{aligned} \chi^{0,m}_{p,q}(j,t) &= 2t^{-\frac{1}{8}} t^{-h^{0}_{m}(p,q)} \chi^{0,m}_{p,q}(t) \chi^{0}(t) p(t)^3 \\ &\times \sum_{r \in I\!N - \frac{1}{2}} \sum_{k \in I\!N} (-1)^{k-1} t^{\frac{1}{2}(r^2 + k(k-1))} (1 - t^k)(1 - t^{r + \frac{1}{2}})(t^{k|j-r|} - t^{k(j+r+1)}) \end{aligned}$$

where in the case of $h^0_m(p,q) = 0$ the multiplicity of 0-level, spin $\frac{1}{2}$ representations should be corrected by 2. The GSO projection in this sector removes half of the representations at each excited level [8].

In the Neveu-Schwarz sector $j \in I\!N$, and

$$\begin{aligned}\chi^{1,m}_{p,q}(j,t) &= t^{-h^1_m(p,q)}\chi^{1,m}_{p,q}(t)\chi^1(t)p(t)^3 \\ &\quad \times\sum_{r\in I\!N}\sum_{k\in I\!N}(-1)^{k-1}t^{\frac{1}{2}(r^2+k(k-1))}(1-t^k)(1-t^{r+\frac{1}{2}})(t^{k|j-r|}-t^{k(j+r+1)})\end{aligned}$$

The GSO projection removes all integer levels. The spin content of the resulting model can be read of from coefficients in front of the half-integer powers in $\chi^{1,m}_{p,q}(j,t)$.

The fermionic closed string Hilbert space can be constructed as the subspace of the tensor product $\mathcal{H}^{(\text{left})\epsilon}_m(p,q)\otimes\mathcal{H}^{(\text{right})\widetilde{\epsilon}}_{\widetilde{m}}(\widetilde{p},\widetilde{q})$ of two open string Hilbert spaces determined by the conditions

$$a^i_0 = \widetilde{a}^i_0 = \frac{P^i_c}{2\sqrt{\alpha}} \quad , \quad a^+_0 = \widetilde{a}^+_0 = \frac{P^+_c}{2\sqrt{\alpha}} \quad , \quad c_0 = \widetilde{c}_0 = 0 \quad ,$$

and anihillated by the twist operator $L^{T\,(\text{left})}_0 - L^{T\,(\text{right})}_0$. Although more general heterotic-like constructions are possible we shall restrict ourselves to the models with $m = \widetilde{m}$. We also assume the GSO projection in the left, and in the right sector, separately.

Let $m \geq 2$ and $\epsilon = \widetilde{\epsilon}$. In this case the space-time spectrum contains only integer spins. Possible models are restricted by the twist condition

$$|h^\epsilon_m(p,q) - h^\epsilon_m(\widetilde{p},\widetilde{q})| \in I\!N_0$$

which admits obvious diagonal solutions $p = \widetilde{p}, q = \widetilde{q}$. One can check by numerical calculations that for sufficiently large m there are also some non-diagonal solutions in both sectors.

The half-integer spin spectrum is possible only for the "mixed" sectors $\epsilon + \widetilde{\epsilon} = 1$. In this case the twist condition takes the form

$$|h^1_m(p,q) + \frac{1}{2} - h^0_m(\widetilde{p},\widetilde{q})| \in I\!N_0$$

where the factor $\frac{1}{2}$ comes from the GSO projection. One can easily show that there are no solutions for odd m. Numerical calculations suggest that there are always some solutions for even m, but we have not succeeded in finding a simple proof.

As in the case of bosonic string the closed string character generating function can be obtained as the diagonal part of the product of the left, and the right open string generating functions.

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